A Note on Arithmetical Properties of Multiple Zeta Values

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Multiple zeta values

$$\zeta(s_1, s_2, \ldots, s_l) = \sum_{n_1 > n_2 > \cdots > n_l \geq 1} \frac{1}{n_1^{s_1} \cdots n_l^{s_l}}, \quad s_j \in \mathbb{N}$$

are actively studied, but the majority of results are various identities between these values. In this paper we touch their arithmetical properties.

The *weight* of the vector $\vec{s} = (s_1, s_2, \ldots, s_l)$ (or of the series $\zeta(s_1, s_2, \ldots, s_l)$) is the sum $s_1 + s_2 + \cdots + s_l$. Let $\mathbb{D}_w$ be the dimension (over $\mathbb{Q}$) of the $\mathbb{Q}$-vector space generated by all multiple zeta values of weight $w \geq 2$.

The following conjecture on $\mathbb{D}_w$ appears in [1].

Conjecture 1 *If $w \geq 2$, then $\mathbb{D}_w = d_w$, where the sequence $\{d_w\}$ is given by the recurrence $d_w = d_{w-3} + d_{w-2}$ with initial values $d_0 = 1$, $d_1 = 0$, $d_2 = 1$.*

The numbers d_w can be defined by the generating function

$$\sum_{w=0}^{\infty} d_w x^w = \frac{1}{1 - x^2 - x^3}.$$

By Conjecture 1 the number of linearly independent numbers among the 2^{w-2} multiple zeta values of weight w is $d_w = O(\alpha^w)$ as $w \to \infty$, where $\alpha = 1.32..$ is the root of $x^3 - x - 1 = 0$. So there should be a lot of (independent) relations among these values. Having certain relations, we can prove an upper estimate for $\mathbb{D}_w$. The paper [2] claims to prove the inequality $\mathbb{D}_w \leq d_w$. However, the proof is very complicated.

A lower estimate for $\mathbb{D}_w$ is connected with the arithmetical nature of multiple zeta values. With the help of multiplication by $\zeta(w_1 - w_2)$, it is easy to show that $\mathbb{D}_{w_1} \geq \mathbb{D}_{w_2}$ if $w_1 \geq w_2 + 2$. However, it is not proved that $\mathbb{D}_w > 1$ for a single w!

Choose the following sets among all vectors with positive integer components:

$$\mathcal{B} = \{\vec{s} : s_i \in \{2, 3\}\}, \quad \mathcal{B}_w = \{\vec{s} \in \mathcal{B} : w(\vec{s}) = w\}.$$

M. Hoffman ([3]) made the following conjectures.

Conjecture 2 *For any $\vec{s_0}$ the value $\zeta(\vec{s_0})$ can be represented as a linear form with rational coefficients in values $\zeta(\vec{s})$ with $\vec{s} \in \mathcal{B}_{w(\vec{s_0})}$.*

This conjecture has been checked by Hoang Ngoc Minh for $\vec{s_0}$ with weight ≤ 16.

Conjecture 3 *If $\vec{s_1}$, ..., $\vec{s_n}$ are distinct elements of $\mathcal{B}$, then 1, $\zeta(\vec{s_1})$, ..., $\zeta(\vec{s_n})$ are linearly independent over $\mathbb{Q}$.*

Suppose that Conjecture 3 is true; then the linear form representation from Conjecture 2 is unique. From Conjectures 2 and 3 there follows Conjecture 1 (see [3]).

Since $\zeta(\{2\}_k) = \frac{\pi^{2k}}{(2k+1)!}$ (see [4, (36)]), these values are irrational (moreover they and 1 are linearly independent over $\mathbb{Q}$). Also, by Apery's theorem, the number $\zeta(3)$ is irrational. There is no certainty about the arithmetic nature of $\zeta(\vec{s})$ for any other $\vec{s} \in \mathcal{B}$.

Let $\zeta(\vec{s_0}) \in \mathbb{Q}$ and $w(\vec{s_0})$ be odd. Suppose that $\zeta(\vec{s_0})\zeta(2k)$ is represented as a linear combination with rational coefficients in numbers $\zeta(\vec{s})$, $\vec{s} \in \mathcal{B}_{w(\vec{s_0})+2k}$ (as implied by Conjecture 2); then there is at least one irrational among these numbers. For instance if $\zeta(2,3) \in \mathbb{Q}$ or $\zeta(3,2) \in \mathbb{Q}$ then one of the numbers $\zeta(3,2,2)$, $\zeta(2,3,2)$ and $\zeta(2,2,3)$ is irrational. Similarly, suppose that $\zeta(\vec{s_0}) \in \mathbb{Q}$, $w(\vec{s_0})$ is even and $\zeta(\vec{s_0})\zeta(3)$ is represented as a linear combination with rational coefficients in numbers $\zeta(\vec{s})$, $\vec{s} \in \mathcal{B}_{w(\vec{s_0})+3}$; then there is at least one irrational among them.

Furthermore, we prove a certain result about the linear independence of multiple zeta values of different weights.

Lemma 1 *Let $x \notin \mathbb{Q}$, numbers y_i, $i = 1, \ldots, k$ be such that 1, y_1, ..., y_k are linearly independent over $\mathbb{Q}$. Then there exist $k-1$ numbers among the xy_i such that 1, x and they are linearly independent over $\mathbb{Q}$.*

Proof. We prove by contradiction. Let the numbers $1, x, xy_1, \ldots, xy_{k-1}$ be linearly dependent over $\mathbb{Q}$. I.e. there exist integers A_1, B_1 and C_{1i} not all zero such that

$$A_1 + B_1 x + \sum_{i=1}^{k-1} C_{1i} x y_i = 0.$$

If $A_1 = 0$ then divide this equality by x; we get that 1, y_1, …, y_{k-1} are linearly dependent, which contradicts the hypothesis. If all $C_{1i} = 0$ then x is rational. Hence there exists $p \in [1, k-1]$ such that $C_{1p} \neq 0$. Let integers A_2, B_2 and C_{2i} be not all zero such that

$$A_2 + B_2 x + \sum_{1 \le i \le k, i \neq p} C_{2i} x y_i = 0.$$

Similarly $A_2 \neq 0$. Multiply the first equality by A_2 and subtract the second equality, multiplied by A_1. We get (letting $C_{1k} = 0$, $C_{2p} = 0$)

$$(B_1 A_2 - B_2 A_1) x + \sum_{i=1}^{k} (C_{1i} A_2 - C_{2i} A_1) x y_i = 0.$$

Divide this equality by x. Then we get a linear form in 1, y_1, …, y_k; moreover the coefficient of y_p is $C_{1p} A_2 \neq 0$, which contradicts the linear independence of 1 and the numbers y_i. The lemma is proved.

Corollary 1 *For any positive integer l, the numbers 1, $\zeta(3)$ and certain l numbers from $\zeta(3)\zeta(2k)$, $k = 1, \ldots, l+1$ are linearly independent over $\mathbb{Q}$.*

Proof. Take $x = \zeta(3)$, $y_k = \zeta(2k)$ in Lemma 1.

This corollary and the equality

$$\zeta(3)\zeta(2k) = \zeta(2k+3) + \zeta(3, 2k) + \zeta(2k, 3)$$

yield another

Corollary 2 *For each $w \ge 2$, let $\mathcal{M}_w$ be a set of vectors of weight w such that all multiple zeta values of the weight w are rational linear combinations of $\zeta(\vec{s})$ with $\vec{s} \in \mathcal{M}_w$. Then for any positive integer l, there exists a subset I of $\{5, 7, \ldots, 2l+5\}$ with $\#I = l$, and l vectors $\vec{t_i} \in \mathcal{M}_i$, i ranging over I, such that 1, $\zeta(3)$ and the numbers $\zeta(\vec{t_i})$ are linearly independent over $\mathbb{Q}$.*

From the equality $\zeta(\{2\}_k) = \frac{\pi^{2k}}{(2k+1)!}$ it is clear that

$$\dim_{\mathbb{Q}}(\mathbb{Q} \oplus \bigoplus_{\vec{s} \in \mathcal{B}_2 \cup \mathcal{B}_4 \cup \cdots \cup \mathcal{B}_{2l}} \mathbb{Q}\zeta(\vec{s})) \ge l + 1.$$

By Conjecture 2 it is possible to take $\mathcal{B}_w$ as the set $\mathcal{M}_w$ required in Corollary 2. If so, then

$$\dim_{\mathbb{Q}}(\mathbb{Q} \oplus \bigoplus_{\vec{s} \in \mathcal{B}_3 \cup \mathcal{B}_5 \cup \cdots \cup \mathcal{B}_{2l+5}} \mathbb{Q}\zeta(\vec{s})) \geq l+2.$$

Since Conjecture 2 has been checked for vectors of weights ≤ 16, this estimate is not conditional for $l \leq 5$.

Corollary 3 *There exists*

$$\vec{s_0} \in \{(2,3),(3,2),(2,2,3),(2,3,2),(3,2,2)\},$$

such that the numbers 1, $\zeta(3)$ and $\zeta(\vec{s_0})$ are linearly independent over $\mathbb{Q}$.

Proof. We use Corollary 2 with $l = 1$ choosing $M_5 = \{(2,3),(3,2)\}$ and $M_7 = \{(2,2,3),(2,3,2),(3,2,2)\}$.

We emphasize that the result of Corollary 3 is unconditional.

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References

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